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Implementing 2 → *M* Phase-covariant Cloning in Spin Networks

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Abstract We propose a scheme to implement $2 \to M$ phase-covariant quantum cloning machine by using a $M+2$ spin star network in which the two central spins interact with the M outer spins respectively. The effect of the magnetic field on the fidelity of the cloning is also investigated. By applying an external magnetic field on the spin system, the fidelity of the cloning machine can be largely improved.

Keywords Quantum cloning · Spin network · Fidelity · Magnetic field

1 Introduction

Quantum information processing (QIP) is derived from the combination of quantum mechanics and information technology [1]. In recent years, a large amount of promising applications of QIP have been implemented theoretically and experimentally, such as quantum cryptography [2], quantum cloning [3, 4], quantum communication [5], quantum entanglement [6, 7] and so on. Among those, quantum closing has attracted a great deal of attention because it distribute the quantum information of an initial input state onto multiple output states. However, the no-cloning theorem, one of the cornerstones of quantum information, prohibits accurate cloning for an arbitrary state [8]. So much attention has been paid to approximate copying [3, 4]. Bužek and Hillery proposed an optimal $1 \to 2$ universal quantum cloning machine (UQCM) with the fidelity of $5/6$ which is input-state independent [3]. Later, Gisin and Massar presented the unitary transformation of $1 \to M$ universal quantum cloning machine [4], and Bruß et al. proposed the phase-covariant cloning machine (PCCM) which is state-dependent [9]. These cloning machines mentioned have only one input-state,

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which are less efficient than the QCM's proper, so the most general quantum cloning machine is the $N \to M$ quantum cloning machine which consists of N input qubits all in the same quantum state, $M - N$ "blank" qubits and K "auxiliary" qubits [10]. It's fidelity is usually dependent on the number N of the input states and the number M of the output copies. Until now, several protocols for implementing cloning machines have been already achieved experimentally [11–15]. Most of these proposals are based on quantum gates which should consider the post-selection of the state [16, 17].

Currently, the solid-state system, especially the spin chain has attracted a great deal of attention because the spin-system performs a free dynamical evolution without almost any external control [18–20]. Moreover, it can be isolated from the environment more efficiently than the traditional method based on quantum logic circuits. The scheme for PCCM based on spin chains has been realized [21, 22], then Chen and Zhang et al. further generated it to the $1 \to M$ phase-covariant cloning and $1 \to M$ universal quantum cloning, respectively [23, 24]. Stimulated by these works, in this paper, we show that by properly choosing the topology of the spin-network and the initial state of the cloning machine, a $2 \to M$ quantum cloning machine can be realized via the free evolution of the spin system and its fidelity can be improved by applying an external magnetic field.

The paper is organized as follows. In Sect. 2 we describe a dynamical evolution that allow us to construct a $2 \to M$ phase-covariant cloning machine. In Sect. 3 we apply an external magnetic field on the original system and study the effect of the magnetic field on the fidelity of quantum cloning. The paper finally ends with a discussion and summary.

2 Implementation of $2 \to M$ PCCM

The spin network involoved in our scheme forms a star configuration(as shown in Fig. 1). The spins labelled 0 and 1 are two central spins, which interact respectively with the outer spins around them and there is no direct coupling among the outer spins. We consider the Heisenberg XX model without any externally applied magnetic field. The Hamiltonian is given by

$$H = \mathcal{J}\left(\sigma_0^x \sum_{i=2}^{N+1} \sigma_i^x + \sigma_0^y \sum_{i=2}^{N+1} \sigma_i^y\right) + \mathcal{J}\left(\sigma_1^x \sum_{i=2}^{N+1} \sigma_i^x + \sigma_1^y \sum_{i=2}^{N+1} \sigma_i^y\right), \tag{1}$$

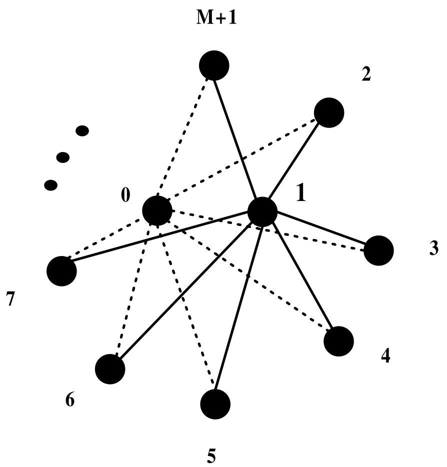

Fig. 1 Spin star network for $2 \to M$ quantum cloning machine. The central spins 0 and 1 interact with all the other spins exclusively and the spins $2 - N$ represent the outer spins

where $\sigma_0^{x,y}, \sigma_1^{x,y}, \sigma_i^{x,y}$ $(i = 2, \ldots, N+1)$ are corresponding to the first central qubit, the second central qubit and the ith outer spin, respectively. $\mathcal{J}$ is the exchange coupling between the central qubits and the outer spins. We note that the outer spins collectively behave as a single spin $\boldsymbol{J} = \hat{i} J_x + \hat{j} J_y + \hat{k} J_z$ $(J_x = \frac{1}{2}\sum_{i=2}^{N+1} \sigma_i^x, J_y = \frac{1}{2}\sum_{i=2}^{N+1} \sigma_i^y)$ which commutes with H. Using the raising and lowering operators $\sigma_\pm = (\sigma_x + i\sigma_y)$ and $J_\pm = \frac{1}{2}\sum_{i=2}^{N+1} \sigma_\pm$, it is convenient to transform the Hamiltonian of (1) to the following form

$$H = \mathcal{J}(\sigma_{0+} J_- + \sigma_{0-} J_+) + \mathcal{J}(\sigma_{1+} J_- + \sigma_{1-} J_+). \tag{2}$$

Inspired by the above form of H which is similar to the Jaynes-Cummings model of quantum optics [25], the Hamiltonian can be considered as two spins 1/2 particles resonantly interact with a higher spin respectively [26].

The initial state of the system is chosen as

$$|\psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)_0 \otimes (\alpha|0\rangle + \beta|1\rangle)_1 \otimes |000\cdots 00\rangle_{2\cdots N}, \tag{3}$$

where the first two kets denote the input states to be copied, respectively, with $\alpha = \cos\frac{\theta}{2}, \beta = e^{i\varphi}\sin\frac{\theta}{2}$ and the state $|000\cdots 00\rangle$ represents the state of the outer spins all in the same direction which can be described as $|j, -j\rangle$ [26]. The output state can be conjectured from the Hamiltonian in (2) as the following form

$$\begin{aligned}
|\psi(t)\rangle &= \alpha^2|00\rangle|j,-j\rangle + a(t)|01\rangle|j,-j\rangle + b(t)|10\rangle|j,-j\rangle + c(t)|00\rangle|j,-j+1\rangle \\
&\quad + d(t)|11\rangle|j,-j\rangle + e(t)|01\rangle|j,-j+1\rangle + f(t)|10\rangle|j,-j+1\rangle \\
&\quad + g(t)|00\rangle|j,-j+2\rangle.
\end{aligned} \tag{4}$$

Notice that the excitation number conserves among the three eigenstates $|01\rangle|j,-j\rangle$, $|10\rangle|j,-j\rangle$, $|00\rangle|j,-j+1\rangle$ and the four eigenstates $|11\rangle|j,-j\rangle$, $|01\rangle|j,-j+1\rangle$, $|10\rangle|j,-j+1\rangle$, $|00\rangle|j,-j+2\rangle$ respectively, applying the Schrödinger equation, we obtain the two equation sets $(M = 2j)$

$$\begin{aligned}
i\dot{c}(t) &= \sqrt{M}[a(t) + b(t)], \\
i\dot{a}(t) &= \sqrt{M}c(t), \\
i\dot{b}(t) &= \sqrt{M}c(t),
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
i\dot{d}(t) &= \sqrt{M}(e(t) + f(t)), \\
i\dot{e}(t) &= \sqrt{M}d(t) + \sqrt{2(M+1)}g(t), \\
i\dot{f}(t) &= \sqrt{M}d(t) + \sqrt{2(M+1)}g(t), \\
i\dot{g}(t) &= \sqrt{2(M+1)}[e(t) + f(t)],
\end{aligned} \tag{6}$$

with the initial condition

$$\begin{aligned}
&a(0) = \alpha\beta, \quad b(0) = \alpha\beta, \quad c(0) = 0, \quad d(0) = \beta^2, \quad e(0) = 0, \\
&f(0) = 0, \quad g(0) = 0,
\end{aligned} \tag{7}$$

where we have assume the spin coupling $\mathcal{J} = 1$.

The solution of the two sets are obtained respectively as

$$a(t) = \frac{1}{2}e^{i\varphi}\cos\sqrt{2M}t,$$
$$b(t) = \frac{1}{2}e^{i\varphi}\cos\sqrt{2M}t, \tag{8}$$
$$c(t) = -\frac{i}{\sqrt{2}}e^{i\varphi}\sin\sqrt{2M}t,$$

and

$$d(t) = \frac{e^{2i\varphi}}{6M-4}(-2+2M+M\cosh\sqrt{4-6M}t),$$
$$e(t) = -\frac{ie^{2i\varphi}}{2\sqrt{4-6M}}(\sqrt{M}\sinh\sqrt{4-6M}t),$$
$$f(t) = -\frac{ie^{2i\varphi}}{2\sqrt{4-6M}}(\sqrt{M}\sinh\sqrt{4-6M}t), \tag{9}$$
$$g(t) = \frac{\sqrt{2}e^{2i\varphi}}{3M-2}\sqrt{M(M-1)}\sinh\left(\sqrt{1-\frac{3M}{2}}t\right)^2.$$

In order to evaluate the performance of our cloning machine, we should calculate the fidelities of both the central spins and the outer spins. For the reason of symmetry, we only need to calculate the reduced density matrix of the central qubit "0" and one of the outside qubits.

The reduced density matrix of the central qubit "0" is

$$\rho_0(t) = \begin{pmatrix} |\alpha^2|^2 + |a(t)|^2 + |c(t)|^2 + |e(t)|^2 + |g(t)|^2 & \alpha^2 b^*(t) + a(t)d^*(t) + c(t)f^*(t) \\ \alpha^2 b(t) + d(t)a^*(t) + f(t)c^*(t) & |b(t)|^2 + |d(t)|^2 + |f(t)|^2 \end{pmatrix}, \tag{10}$$

and corresponding fidelity is

$$\begin{aligned} F_0 &= \langle\psi_0|\rho_0(t)|\psi_0\rangle \\ &= |\alpha|^2(|\alpha|^2 + |a(t)|^2 + |c(t)|^2 + |e(t)|^2 + |g(t)|^2) \\ &\quad + \alpha^*\beta(\alpha^2 b^*(t) + a(t)d^*(t) + c(t)f^*(t)) \\ &\quad + \beta^*\alpha(\alpha^2 b(t) + d(t)a^*(t) + f(t)c^*(t)) \\ &\quad + |\beta|^2(|b(t)|^2 + |d(t)|^2 + |f(t)|^2). \end{aligned} \tag{11}$$

For the equatorial states ($\theta = \pi/2$), $\alpha = \frac{\sqrt{2}}{2}$, $\beta = \frac{\sqrt{2}}{2}e^{i\varphi}$, substituting all of the coefficients in (8) and (9) into (11), we can easily get

$$\begin{aligned} F_0 = \frac{1}{2} + \frac{1}{4}\Bigg(&\cos\sqrt{2M}t + \frac{\cos\sqrt{2M}t(-2+2M+M\cosh\sqrt{4-6M}t)}{3M-2} \\ &+ \frac{\sqrt{M}\sin\sqrt{2M}t\sinh\sqrt{4-6M}t}{\sqrt{2-3M}}\Bigg). \end{aligned} \tag{12}$$

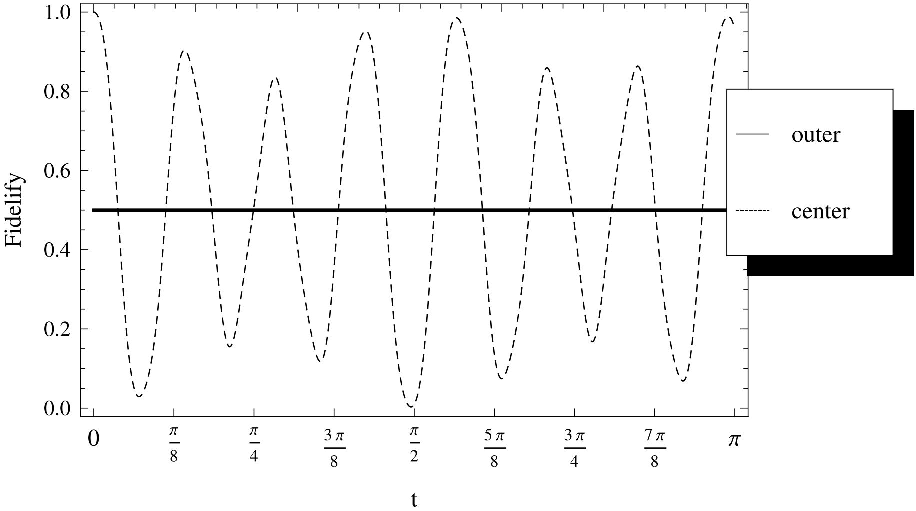

Fig. 2 The fidelities of the central spins and the outer spins without any outer applied magnetic fields for $M = 4$

Similarly we obtain the reduced density matrix of one of the outer qubits

$$\rho_{outer} = \begin{pmatrix} |\alpha^2|^2 + |a(t)|^2 + |b(t)|^2 + \frac{M-1}{M}|c(t)|^2 & \frac{1}{\sqrt{M}}[\alpha^2 c^*(t) + a(t)e^*(t) + b(t)f^*(t)] \\ +\frac{M-1}{M}|e(t)|^2 + \frac{M-1}{M}|f(t)|^2 + \frac{M-2}{M}|g(t)|^2 & +\frac{\sqrt{2(M-1)}}{M}c(t)g^*(t) \\ \frac{1}{\sqrt{M}}[\alpha^2 c(t) + a^*(t)e(t) + b^*(t)f(t)] & \frac{1}{M}[|c(t)|^2] + |e(t)|^2 + |f(t)|^2] \\ +\frac{\sqrt{2(M-1)}}{M}c^*(t)g(t) & +\frac{2}{M}|g(t)|^2 \end{pmatrix}, \tag{13}$$

and the fidelity of the outer spins for the equatorial states is

$$F_{outer} = \langle\psi_0|\rho_{outer}(t)|\psi_0\rangle = \frac{1}{2}. \tag{14}$$

It is obvious that both fidelities of central and outer spins do not depend on φ, and the fidelity of the outer spins is a coefficient regardless of the value of M. Both evolutions of the fidelities of central and outer spins are shown in Fig. 2.

3 The Effect of the Magnetic Field

In order to improve the fidelity of outer spins, we apply an external magnetic field B which preserve z component of the spin angular moment of the system on the above XX model in this section. The corresponding Hamiltonian is written as

$$H = (\sigma_{0+}J_- + \sigma_{0-}J_+) + (\sigma_{1+}J_- + \sigma_{1-}J_+) + B(S_0^z + S_1^z) + BJ^z, \tag{15}$$

where $J^z = \sum_{outer}\sigma_i^z/2$, $s^z = \sigma^z/2$ and B corresponds to the external magnetic fields which were applied on the central qubits and the outer spins respectively. In this case, we can modulate this field in order to improve the fidelity of quantum cloning machine.

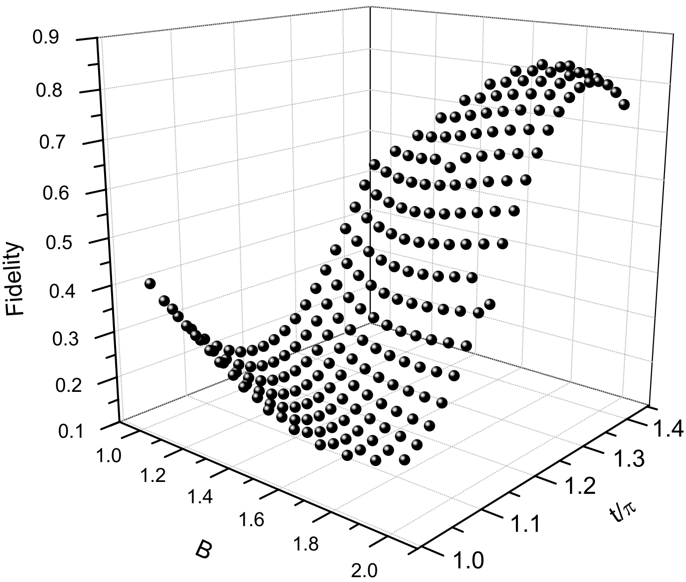

Fig. 3 The fidelities of the outer spins under the applied magnetic field for $M = 4$, $B = 1.0, 1.1, 1.2, \dots, 2.0$

The output state of the system can be expressed as:

$$
\begin{aligned}
|\psi'(t)\rangle &= a'(t)|00\rangle|j,-j\rangle + b'(t)|01\rangle|j,-j\rangle + c'(t)|10\rangle|j,-j\rangle + d'(t)|00\rangle|j,-j+1\rangle \\
&\quad + e'(t)|11\rangle|j,-j\rangle + f'(t)|01\rangle|j,-j+1\rangle + g'(t)|10\rangle|j,-j+1\rangle \\
&\quad + h'(t|00\rangle|j,-j+2\rangle).
\end{aligned}
\tag{16}
$$

By the similar calculations, we get the reduced density operator of an outer spin

$$
\rho'_{outer} = \begin{pmatrix}
\begin{array}{c} |a'(t)|^2 + |b'(t)|^2 + |c'(t)|^2 + |e'(t)|^2 + \frac{M-1}{M}|d'(t)|^2 \\ + \frac{M-1}{M}|f'(t)|^2 + \frac{M-1}{M}|g'(t)|^2 + \frac{M-2}{M}|h'(t)|^2 \end{array} & \begin{array}{c} \frac{1}{\sqrt{M}}[a'(t)d'^*(t) + b'(t)f'^*(t) + c'(t)g'^*(t)] \\ + \frac{\sqrt{2(M-1)}}{M} d'(t)h'^*(t) \end{array} \\
\begin{array}{c} \frac{1}{\sqrt{M}}[a'^*(t)d'(t) + b'^*(t)f'(t) + c'^*(t)g'(t)] \\ + \frac{\sqrt{2(M-1)}}{M} d'^*(t)h'(t) \end{array} & \begin{array}{c} \frac{1}{M}[|d'(t)|^2] + |f'(t)|^2 + |g'(t)|^2] \\ + \frac{2}{M}|h'(t)|^2 \end{array}
\end{pmatrix},
\tag{17}
$$

where all the coefficients are connected with B and M. The fidelity of the outer spins with magnetic field can be obtained as

$$
F'_{outer} = \langle\psi_0|\rho'_{outer}(t)|\psi_0\rangle.
\tag{18}
$$

However, it's difficult to get the analytic expressions of the fidelity, so without loss of generality, we use the numerical calculations for analyzing the fidelity of cloning. The fidelities of the outer spins under different magnetic field are shown in Fig. 3.

4 Discussion and Summary

The fidelity stands for the quality of cloning, so we briefly analyze the fidelities of the quantum cloning machine in different situations. The optimal fidelity of $N \to M$ $(N < M)$

quantum cloning machine is [27]

$$F_{opt} = \frac{M(N+1)+N}{M(N+2)}. \tag{19}$$

From Fig. 2 we can see that the fidelity of the outer spins $\frac{1}{2}$ is far less than the corresponding optimal fidelity $\frac{7}{8}$ of $N \to M$ quantum cloning machine. However, we can easily find from Fig. 3 that the fidelity of the outer spins appears an oscillation with B and M. We chose $B = 1.8$ and $t = 1.38\pi$, the fidelity can reach 0.825 which is very close to the optimal fidelity and obviously higher than the fidelity without the magnetic field. It means that the external magnetic field plays a very important role in improving the fidelity of the quantum cloning machine.

In summary, we have proposed a scheme to implement $2 \to M$ PCCM basing on a spin network. It requires that the two central spins interact with all the outer spins, respectively. By analyzing the fidelities of the outer spins with or without outer magnetic field we find that the fidelity of the quantum cloning machine can be largely improved by applying an external magnetic field. Moreover, the interaction is based on the spin couplings which perform a free dynamical evolution without almost any external control thus the influence of the environment can be neglect.

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